Packet Error Probabilities in Direct-Sequence Spread-Spectrum Packet Radio Networks

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Abstract—**The problem of computing packet error probabilities in direct-sequence spread-spectrum packet radio networks is difficult. Packet errors are caused by a combination of noise at the receivers and interference between packet transmissions, which overlap in time. The interference between packet transmissions produces dependent errors at the output of the demodulator. In this paper, we compute an upper bound on the packet error probability induced in direct-sequence spread-spectrum networks. The bound, which we introduce, is valid independently of whether signals arrive with equal or unequal powers at the receiver site. Furthermore, it has a simple form and it is easy to compute. Finally, it is valid for various classes of forward error correction codes (e.g., BCH, convolutional codes). In this paper, numerical results are presented for BCH codes only.**

I. Introduction

The rapid growth of computer communication has motivated an intense interest in packet switching radio techniques ([1]). Furthermore, there is a growing need for computer communication and information distribution in tactical military applications where spread-spectrum waveforms must be used in order to achieve reliable operation in the presence of intentional interference (jamming). As a result, a thorough investigation of spread-spectrum packet radio networks becomes necessary.

An important attribute of spread-spectrum signaling is its multiple-access capability ([4]). In our work, we are going to examine the multiple-access capability of direct-sequence spread-spectrum packet radio networks. The most important indicator of the multiple-access capability of a packet radio network is the induced packet error probability.

The problem of computing packet error probabilities in direct-sequence spread-spectrum packet radio networks is difficult. Packet errors are caused by a combination of noise at the receivers and interference between packet transmissions. which overlap in time. The interference between packet transmissions produces dependent errors at the output of the demodulator. A lot of work has been directed towards the evaluation of the bit error probability in direct sequence spread spectrum networks ([2], [3], [10]). The dependency of the bit errors does not allow us to extend the results in [2] and [3], in order to compute the packet error probability.

To the best of our knowledge, the first serious effort to compute packet error probabilities in direct-sequence spread-spectrum networks was conducted in [5]. In [5], the authors compute an upper bound on the packet error probability induced in a direct-sequence spread-spectrum packet radio network, which utilizes binary convolutional coding, hard-decision demodulation, Viterbi decoding and random signature sequences.

The upper bound on the packet error probability, derived in [5], has been proven to be valid only when the signals arrive with equal power at the receiver site. This is a severe limitation because, in general, signals arrive at the receiver site with unequal powers. In this paper, utilizing the Chernoff bound, we find an upper bound on the packet error probability induced in direct-sequence spread-spectrum packet radio networks, when BCH codes are used for the encoding of the packets. Our bound is valid independently of whether signals arrive with equal or unequal powers at the receiver site. The upper bound, which we introduce, has a simple form and it is easy to compute. Furthermore, as we show in Section IV, it can be improved at the expense of increased computational complexity. In addition to that, it is valid for other classes of codes (e.g., convolutional codes), but in this paper numerical results are presented for BCH codes only.

The organization of the paper is as follows. In Section II, the model and some preliminary facts are provided. In Section III, the upper bound on the packet error probability is presented, accompanied by the numerical results. Comparisons between our bound and the bound presented in [5] are included in Section IV. In the same section, a method to improve our bound is described. Finally, in Section V, we make some conclusive remarks.

II. The Model-Preliminaries

The model for direct-sequence spread-spectrum transmission considered here is described in [6]. The only difference is that the signature sequence is assumed to be a sequence of independent, identically distributed, binary random variables, each equally likely to be $+1$ or -1. Each transmitter in the network has such a sequence, and each sequence is assumed to be independent of the sequences of other transmitters.

Let us now assume that we have a slotted channel (i.e., packet transmissions initiate at the beginnings of slots), K ($K > 1$) packet transmissions occur within a slot, and a receiver locks on to packet #1 (packets are indexed #1, #2, $\cdots$, #K). Each packet originates from a different transmitter in the network. A packet is exactly one codeword from a (M, L) BCH code (M = total number of codeword bits, L = total number of information bits; the bits of a codeword are indexed from 0 up to $M-1$).

Our objective is to compute the probability that the receiver decodes packet #1 incorrectly. We denote this probability by $P_e(K)$. It is worth noting that $P_e(K)$ is an upper bound on the probability that packet #1 is incorrectly decoded by the receiver in an unslotted channel, provided that $K-1$ corresponds to the maximum number of packets interfering with packet #1.

The receiver is assumed to be a correlation receiver. The output of the receiver, corresponding to the mth bit ($0 \le m \le M-1$) of packet #1, is the random variable (see [4] for more details)

$$Z_m = n_m + \left(2^{-1}P_1\right)^{1/2} T \left\{ b_m^{(1)} + \sum_{i=2}^{K} \left(P_i/P_1\right)^{1/2} I_{i,1}^m\left(b_i^m, \tau_i, \phi_i\right) \right\}; \quad 0 \le m \le M-1. \quad (1)$$

Each n_m is a Gaussian random variable with zero mean and variance $N_0T/4$ where $N_0/2$ is the two sided spectral density of the white Gaussian noise and T is the data bit duration. The random variables n_m ($0 \le m \le M-1$) are independent. The variable $b_m^{(1)}$ represents the mth bit of packet #1; its value is either $+1$ or -1. The vector b_i^m represents a pair of consecutive data bits of packet

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$\# i$. In particular, $b_i^m = (b_{m-1}^{(i)}, b_m^{(i)})$, and each data bit $b_m^{(i)}$ is either $+1$ or -1. Each τ_i or ϕ_i is a random variable representing the time delay (modulo T) or the phase angle (modulo 2π), respectively, of packet $\# i$ relative to packet $\# 1$. As in [4], we take the range of τ_i to be the interval $[0, T]$ and the range of ϕ_i to be the interval $[0, 2\pi]$. Finally, P_i is the power of packet $\# i$ at the receiver.

The function $I_{i,1}^m$, which appears in (1) represents the normalized multiple-access interference due to packet $\# i$. This function is defined by

$$I_{i,1}^m(b_i^m, \tau, \phi) = T^{-1}\left[b_{m-1}^{(i)} R_{i,1}^m(\tau) + b_m^{(i)} \hat{R}_{i,1}^m(\tau)\right] \cos\phi \quad (2)$$

where the functions $R_{i,1}^m$ and $\hat{R}_{i,1}^m$ are given by

$$R_{i,1}^m(\tau) = \int_{mT}^{mT+\tau} a_i(t-\tau) a_1(t)\, dt \quad (3)$$

$$\hat{R}_{i,1}^m(\tau) = \int_{mT+\tau}^{(m+1)T} a_i(t-\tau) a_1(t)\, dt. \quad (4)$$

Note that the $a_i(t)$ and the $a_1(t)$ in (3) and (4) are the spectral spreading signals corresponding to packets $\# i$ and $\# 1$, respectively. In fact,

$$a_i(t) = \sum_{j=-\infty}^{+\infty} a_j^{(i)} \psi(t - jT_c); \quad 1 \le i \le K \quad (5)$$

where $\{a_j^{(i)}\}$ is the signature sequence corresponding to packet $\# i$, $\psi(t)$ is the chip waveform, and T_c is the chip duration. In this paper, we assume a rectangular chip waveform. Hence,

$$\psi(t) = \begin{cases} 1 & 0 \le t \le T_c \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

The detector decides that the mth bit of packet $\# 1$ is $+1$ or -1 if $Z_m > 0$ or $Z_m < 0$, respectively. It is easy to show that the mth bit of packet $\# 1$ is decoded correctly by the above detector if and only if the random variable

$$X_m = n_m^* + \left[1 + \sum_{i=2}^{K} b_m^{(1)} (P_i/P_1)^{1/2} I_{i,1}^m(b_i^m, \tau_i, \phi_i)\right]; \quad 0 \le m \le M-1 \quad (7)$$

is positive. In (7), each n_m^* is a Gaussian random variable with mean 0 and variance $N_0/2E_b$ where $E_b = P_1 T$ is the energy per data bit of packet $\# 1$. The random variables n_m^* $(0 \le m \le M-1)$ are statistically independent.

Let us now denote by S a random variable, which represents the number of random variables X_m $(0 \le m \le M-1)$ that are negative. Then,

$$P_e(K) = \Pr(S > e) \quad (8)$$

where e corresponds to the error correction capability of the BCH code. We will state two propositions.

Proposition 1: For the computation of $P_e(K)$ the τ_i's $(2 \le i \le K)$ need be known only to the nearest chip.

Proposition 2: $P_e(K)$ is independent of the values of the data bit sequences $\{b_m^{(i)}\}_{m=0}^{M-1}$ for $1 \le i \le K$.

The validity of Propositions 1 and 2 is based on the fact that random signature sequences are utilized. An immediate consequence of Propositions 1 and 2, is that the random variable X_m in (7) assumes the following equivalent form [see also (2)–(6)]:

$$\begin{aligned} X_m = n_m^* + 1 + \sum_{i=2}^{K} (P_i/P_1)^{1/2} \Big\{ & \left[a_{mN-1}^{(i)} a_{mN}^{(1)}\right] \tau_i/T_c \\ & + \left[a_{mN}^{(i)} a_{mN}^{(1)} + a_{mN+1}^{(i)} a_{mN+1}^{(1)} + \cdots + a_{mN+N-1}^{(i)} a_{mN+N-1}^{(1)}\right](1 - \tau_i/T_c) \\ & + \left[a_{mN}^{(i)} a_{mN+1}^{(1)} + a_{mN+1}^{(i)} a_{mN+2}^{(1)} + \cdots + a_{mN+N-2}^{(i)} a_{mN+N-1}^{(1)}\right](\tau_i/T_c)\Big\} \frac{\cos\phi_i}{N}; \\ & 0 \le m \le M-1. \end{aligned} \quad (9)$$

In (9), we assumed that each of the spread-spectrum signals has N chips per bit. Let us make an important observation.

Observation 1: Given the phase (ϕ_i) and the delay (τ_i) of each interfering transmission $(2 \le i \le K)$, the random variables X_m $(0 \le m \le M-1)$ *are not* independent.

To prove our observation we show, in Appendix A, that for $N = 2$, $K = 3$, $\phi_2 = \phi_3 = 0$, $\tau_2 = \tau_3 = T_c/2$, and $P_2/P_1 = P_3/P_1 = 1$ the following inequality is true:

$$\Pr(X_0 < 0 \cap X_1 < 0) \ne \Pr(X_0 < 0)\Pr(X_1 < 0). \quad (10)$$

In some examples of packet radio networks the random variables X_m are conditionally independent (given all delays and phases). Consider, for instance, a packet radio network where each packet consists of two codewords. Each code word is BCH code. We send the first codeword of a packet at bit intervals $0, 2, 4, \cdots$, and the second codeword of the packet at bit intervals $1, 3, 5, \cdots$. Let us denote by S^* the number of random variables X_m $(m = 0, 2, 4, \cdots)$ or the number of random variables X_m $(m = 1, 3, 5, \cdots)$, which are negative. Let us also denote by $P_e^*(K)$ the codeword error probability. Then,

$$P_e^*(K) = \Pr(S^* > e) \quad (11)$$

where e is the error correction capability of the code. The random variables X_m, which affect the codeword error probability in (11), are conditionally independent (given all delays and phases). A loose upper bound on the packet error probability $P_e(K)$, induced in the above packet radio network, is given by the following expression:

$$P_e(K) \le 2P_e^*(K). \quad (12)$$

The above example reveals that appropriate bit interleaving can guarantee the conditional independence of the random variables X_m. In any case, even without bit interleaving, it is the author's belief that the packet error probability $P_e(K)$ will not be severely affected if we treat the random variables X_m as conditionally independent, provided that $K \ll N$. As a result, the derivation of the upper bound on the packet error probability $P_e(K)$, presented in the next section, will be based on the following assumption.

Assumption 1: Given the phase (ϕ_i) and the delay (τ_i) of each interfering transmission $(2 \le i \le K)$, the random variables X_m $(0 \le m \le M-1)$ *are* independent.

III. An Upper Bound on the Packet Error Probability

Let us define the random vectors

$$\begin{aligned} \boldsymbol{\phi} &= (\phi_2, \phi_3, \cdots, \phi_K) \\ \boldsymbol{\tau} &= (\tau_2, \tau_3, \cdots, \tau_K) \end{aligned} \quad (13)$$

Let us denote by $f_{\boldsymbol{\tau},\boldsymbol{\phi}}(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\phi}})$ the joint probability density function of the random vectors $\boldsymbol{\tau}$ and $\boldsymbol{\phi}$. The first step in our effort to compute an upper bound on $P_e(K)$ [see Section II, eq. (8)] is to condition on $\boldsymbol{\phi}$ (all phases) and $\boldsymbol{\tau}$ (all delays). Then, we get

$$P_e(K) = \int_{\hat{\boldsymbol{\tau}}} \int_{\hat{\boldsymbol{\phi}}} \Pr(S > e \mid \boldsymbol{\tau} = \hat{\boldsymbol{\tau}}, \boldsymbol{\phi} = \hat{\boldsymbol{\phi}}) f_{\boldsymbol{\tau},\boldsymbol{\phi}}(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{\phi}})\, d\hat{\boldsymbol{\phi}}\, d\hat{\boldsymbol{\tau}}. \quad (14)$$

Due to assumption 1 and formula (9), we can write

$$\Pr(S > e \mid \boldsymbol{\tau} = \hat{\boldsymbol{\tau}}, \boldsymbol{\phi} = \hat{\boldsymbol{\phi}}) = \sum_{i=e+1}^{M} \begin{bmatrix} M \\ i \end{bmatrix} p^i (1-p)^{M-1} = g(p) \quad (15)$$

where

$$p = \Pr\left(X_0 < 0 \mid \tau = \hat{\tau}, \phi = \hat{\phi}\right) = \Pr\left(n_0^* + 1 + \sum_{i=2}^{K} (P_i/P_1)^{1/2} I_i^0(\hat{\tau}_i, \hat{\phi}_i) < 0\right) \quad (16)$$

with

$$I_i^0(\hat{\tau}_i, \hat{\phi}_i) = \{a_{-1}^{(i)} a_0^{(1)} \hat{\tau}_i/T_c + \left[a_0^{(i)} a_0^{(1)} + \cdots + a_{N-1}^{(i)} a_{N-1}^{(1)}\right](1 - \hat{\tau}_i/\tau_c) + \left[a_0^{(i)} a_1^{(1)} + \cdots + a_{N-2}^{(i)} a_{N-1}^{(1)}\right]\hat{\tau}_i/T_c\} \cdot \cos\hat{\phi}_i/N; \quad 2 \le i \le K. \quad (17)$$

The second step in our work is to find an upper bound p_u on the probability p (i.e., the conditional bit error probability), which is independent of $\hat{\tau}$ and $\hat{\phi}$. By doing so, we can upper bound $P_e(K)$ by $g(p_u)$, since $g(\cdot)$ is an increasing function of its argument [see also (14)].

Let us start by defining the random variable Y, such that

$$Y = n_0^* + \left\{\sum_{i=2}^{K} (P_i/P_1)^{1/2} \hat{I}_i^0(\hat{\tau}_i, \hat{\phi}_i)\right\} - 1 \quad (18)$$

where

$$\hat{I}_i^0(\hat{\tau}_i, \hat{\phi}_i) = \{a_{-1}^{(i)} \hat{a}_0^{(1)} \hat{\tau}_i/T_c + \left[a_0^{(i)} \hat{a}_0^{(1)} + \cdots + a_{N-1}^{(i)} \hat{a}_{N-1}^{(1)}\right](1 - \hat{\tau}_i/T_c) + \left[a_0^{(i)} \hat{a}_1^{(1)} + \cdots + a_{N-2}^{(i)} \hat{a}_{N-1}^{(1)}\right]\hat{\tau}_i/T_c\} \cdot \cos\hat{\phi}_i/N; \quad 2 \le i \le K \quad (19)$$

and $\hat{a}^{(1)}$ corresponds to a fixed choice for the values of the components of the random vector $a^{(1)} = (a_0^{(1)} a_1^{(1)} \cdots a_{N-1}^{(1)})$.

From the total probability formula and the fact that n_0^* and $I_i^0(\tau_i, \phi_i)$ $(2 \le i \le K)$ are symmetric random variables, we conclude that

$$p = \sum_{a^{(1)} = \hat{a}^{(1)}} \Pr(Y \ge 0) \Pr\left(a^{(1)} = \hat{a}^{(1)}\right) \cdot \text{all possible choices of } \hat{a}^{(1)}. \quad (20)$$

We now present two lemmas, which will help us define the upper bound p_u.

Lemma 1:

$$\Pr[Y \ge 0] \le \exp(-z) E[\exp(z n_0^*)] \cdot \prod_{i=2}^{K} E\{\exp[z(P_i/P_1)^{1/2} \hat{I}_i^0(\hat{\tau}_i, \hat{\phi}_i)]\}; \quad z \ge 0 \quad (21)$$

where E denotes the expectation operator.

Lemma 1 is a consequence of the well-known Chernoff bound and the fact that the random variables n_0^* and $\hat{I}_i^0(\hat{\tau}_i, \hat{\phi}_i)$ $(2 \le i \le K)$ are independent. A proof of Chernoff's bound can be found in [7].

Lemma 2:

$$E\{\exp[z(P_i/P_1)^{1/2} \hat{I}_i^0(\hat{\tau}_i, \hat{\phi}_i)]\} \le E\{\exp[z(P_i/P_1)^{1/2} \hat{I}_i^0(0,0)]\}; \quad 2 \le i \le K, \ z \ge 0. \quad (22)$$

A generalized version of Lemma 2 (i.e., Lemma 3, in Section IV) is proven in Appendix B. Its proof is based on the increasing nature of the function $h(t) = e^t + e^{-t}$ $(t \ge 0)$ [i.e., as t increases $(t \ge 0)$, $h(t)$ increases, too].

From Lemmas 1 and 2 we get an upper bound on the $\Pr[Y \ge 0]$, which is independent of $\hat{\tau}$ and $\hat{\phi}$. Furthermore, this bound does not depend on the specific choice of $\hat{a}^{(1)}$. Hence, it is an upper bound on the probability p. Consequently,

$$p_u = \inf_{z \ge 0}\left\{\exp(-z) E[\exp(z n_0^*)] \prod_{i=2}^{K} E[\exp[z(P_i/P_1)^{1/2} J_i]]\right\} \quad (23)$$

with

$$J_i = \left[\sum_{j=0}^{N-1} a_j^{(i)}\right] \Big/ N; \quad 2 \le i \le K. \quad (24)$$

The above discussion allows us to write

$$P_e(K) \le \sum_{i=e+1}^{M} \begin{bmatrix} M \\ i \end{bmatrix} p_u^i (1 - p_u)^{M-i}. \quad (25)$$

Equation (25) provides us with an upper bound on the packet error probability $P_e(K)$. We denote this upper bound $P_e^u(K)$.

In Tables I and II, the upper bound $P_e^u(K)$ on the packet error probability $P_e(K)$ is depicted for specific K values and $N = 31$ or 127 (only the first three most significant digits of $P_e^u(K)$ are shown in the tables). This signal-to-noise ratio (E_b/N_0) is taken to be 12 or 15 dB, and the interfering signals (i.e., packets #2, $\cdots$, #K) are assumed to be 0, 3, or 6 dB stronger than the desired transmission (i.e., packet #1). The results in Tables I and II correspond to the (63, 30), (127, 64), (255, 131), and (1023, 513) BCH codes. The "*" entries in Tables I and II correspond to very small values $(< 10^{-15})$, and the "**" entries correspond to relatively large values (> 0.1).

IV. Comments

Let us first make some comparisons between our bound and already existing bounds. To the best of our knowledge, the only existing upper bounds on $P_e(K)$, in the open literature, correspond to the case when all signals (i.e., packets #1, #2, $\cdots$, #K) arrive at the receiver with equal power (see [5]).

Pursley *et al.* showed in [5] that for equal power signals the probability p [see formula (16)] is upper bounded by the following expression:

$$q = Q\left[(2E_b/N_0)^{1/2}\right] + (1/\pi)\int_0^\infty u^{-1} \sin u \, \phi_2(u)\left[1 - \phi_1(u)\right] du \quad (26)$$

where

$$\phi_1(u) = \{\cos(u/N)\}^{N(K-1)} \quad (27)$$

$$\phi_2(u) = \exp\left[(-N_0/4E_b)u^2\right] \quad (28)$$

and

$$Q(x) = (2\pi)^{-1/2} \int_0^\infty \exp(-u^2/2) \, du. \quad (29)$$

From our discussion in Section II, it follows that we can upper bound the packer error probability $P_e(K)$ by

$$\hat{P}_e^u(K) = \sum_{i=e+1}^{M} \begin{bmatrix} M \\ i \end{bmatrix} q^i (1 - q)^{M-i}. \quad (30)$$

In Table III, we show the values of p_u [see (23)] and q for different K values, when $N = 31$. The signal-to-noise ratio (E_b/N_0) is taken to be 12 or 15 dB and the interfering signals are assumed to be of equal power with the desired transmission. The entries in Table III correspond to the four most significant digits of the quantities p_u and q. Table III indicates that for the same N and E_b/N_0 values and for equal power signals the q bound is 6–11 times better than the p_u bound for various K choices. We have found that this relationship between p_u and q holds for other N

TABLE I
THE UPPER BOUND $P_e^u(K)$ ON THE PACKET ERROR PROBABILITY $P_e(K)$

E_b/N_0=12 , N=31

(1023,513) BCH code

K	0dB	3dB	6dB
2	*	*	8.73D-03
3	*	1.83D-02	**
4	2.32D-12	**	**
5	2.81D-02	**	**
6	**	**	**

(255,131) BCH code

K	0dB	3dB	6dB
2	*	*	9.17D-03
3	2.07D-15	1.36D-02	**
4	9.95D-07	**	**
5	1.73D-02	**	**
6	**	**	**

(127,64) BCH code

K	0dB	3dB	6dB
2	*	3.48D-11	1.52D-02
3	8.17D-11	2.00D-02	**
4	4.64D-05	**	**
5	2.36D-02	**	**
6	**	**	**

(63,30) BCH code

K	0dB	3dB	6dB
2	*	2.29D-08	1.41D-02
3	3.96D-08	1.73D-02	**
4	2.39D-04	**	**
5	1.96D-02	**	**
6	1.83D-01	**	**
7	**	**	**

E_b/N_0=15 , N=31

(1023,513) BCH code

K	0dB	3dB	6dB
2	*	*	5.81D-07
3	*	3.88D-06	**
4	*	**	**
5	1.11D-05	**	**
6	**	**	**

(255,131) BCH code

K	0dB	3dB	6dB
2	*	*	1.23D-04
3	*	2.69D-04	**
4	1.44D-10	**	**
5	4.21D-04	**	**
6	1.69D-01	**	**
7	**	**	**

(127,64) BCH code

K	0dB	3dB	6dB
2	*	*	9.23D-04
3	*	1.51D-03	**
4	2.27D-07	**	**
5	2.02D-03	**	**
6	1.25D-01	**	**
7	**	**	**

(63,30) BCH code

K	0dB	3dB	6dB
2	*	6.90D-12	1.87D-03
3	2.25D-11	2.65D-03	**
4	6.91D-06	**	**
5	3.25D-03	**	**
6	7.47D-02	**	**
7	**	**	**

values as well (e.g., $N = 127$). In Table IV we show $\hat{P}_e^u(K)$ [see (30)] for $E_b/N_0 = 15$ dB, $N = 31$ and for various K values. For the results in Table IV we assumed that a packet consists of a (1023, 513) BCH code and that signals arrive with equal power at the receiver. As in Tables I and II a "*" entry in Table IV corresponds to a very small value ($< 10^{-15}$). Comparing the results of Table I (0 dB case) and IV, we see that $P_e^u(K)$ is very pessimistic for equal power signals. In particular, for the (1023, 513) BCH code $\hat{P}_e^u(K)$ predicts a smaller error probability for 9 users than $P_e^u(K)$ does for 5 users.

Another important issue that should be addressed is the tightness of the upper bound $P_e^u(K)$ presented in Tables I and II. To do so we will compare the upper bound p_u, defined in Section III, to the average bit error probability s induced in our systems. The average bit error probability s is defined as follows.

$$s = \int_{\hat{\tau}} \int_{\hat{\phi}} p f_{\tau,\phi}(\hat{\tau}, \hat{\phi}) \, d\hat{\tau} \, d\hat{\phi} \tag{31}$$

where p is given by expression (16) and we took $f_{\tau,\phi}(\hat{\tau}, \hat{\phi})$ equal to $(2\pi T)^{-(K-1)}$ for all $\hat{\tau}, \hat{\phi}$. In Table V we show upper (s_u) and lower (s_l) bounds on the average bit error probability for $N = 15$; $K = 2, 3$, $N = 31$; $K = 2$, for $E_b/N_0 = 12$ or 15 dB and for near–far ratios (NFR) of 0, 3, 6 dB (see [6] for a method to compute the quantities s_l and s_u). In the same table, we include the ratios r_l and r_u where $r_l = p_u/s_u$ and $r_u = p_u/s_l$. The following observations are pertinent to the results included in Table V.

O.2) The upper bound p_u [and consequently $P_e^u(K)$] is tighter for smaller signal to noise ratios E_b/N_0 (see also [10]).

O.3) For smaller signal-to-noise ratios (e.g., $E_b/N_0 = 12$ dB) the upper bound p_u [and consequently $P_e^u(K)$] is not affected (in terms of orders of magnitude) as the number N of chips per bit increases. For larger signal to noise ratios (e.g., $E_b/N_0 = 15$ dB) the upper bound p_u [and consequently $P_e^u(K)$] deteriorates as N increases.

O.4) The upper bound p_u [and consequently $P_e^u(K)$] is not affected (in terms of orders of magnitude) as K (number of interfering signals) changes.

O.5) The upper bound p_u [and consequently $P_e^u(K)$] is tighter as the power of the interfering signals (near–far ratio) increases.

We would like to point out that 0.2–0.5 should be used with caution, since they are primarily based on the limited number of data points contained in Table V. Note though that observations 2–4 were also based on data points included in [6, Figs. 7, 8, and 9]. Observation 5 allows us to claim that the entries of Tables I and II (although upper bounds) indicate that the performance of direct-sequence spread-spectrum packet radio networks deteriorates rapidly for unequal power signal (the near–far problem). The entries of Table V show that the p_u bound is worse by one and sometimes two orders of magnitude than the average bit error probability. This is another indication that our upper bound $P_e^u(K)$ on the packet error probability is pessimistic.

Let us now concentrate on one possible improvement of the upper bound derived in the previous section. Our starting point is formula (14). Let us make the usual assumptions about $f_{\tau,\phi}$ (i.e., $\tau_2, \tau_3, \cdots, \tau_K, \phi_2, \phi_3, \cdots, \phi_K$ are independent random variables; each τ_i is uniformly distributed in the interval $[0, T_c]$; each ϕ_i is uniformly distributed in the interval $[0, 2\pi]$). Then we can easily express $P_e(K)$ as a multiple integral, similar as the one in (14), but the range of each delay is the interval $[0, T_c/2]$, while the range of each phase is the interval $[0, \pi/2]$ (see [5, Appendix, p. 10] for more details). The improvement of the bound is due to the following generalized version of Lemma 2.

TABLE II
THE UPPER BOUND $P_e^u(K)$ ON THE PACKET ERROR PROBABILITY $P_e(K)$

E_b/N_0=12dB, N=127

(1023,513) BCH code

K	0dB	K	3dB	K	6dB
12	*	6	*	3	*
13	4.79D-13	7	3.18D-13	4	2.30D-13
14	2.95D-09	8	1.88D-06	5	9.81D-03
15	2.54D-06	9	1.11D-02	6	**
16	3.84D-04	10	**		
17	1.31D-02				
18	1.29D-01				

(255,131) BCH code

K	0dB	K	3dB	K	6dB
10	1.87D-13	5	*	3	*
11	7.07D-11	6	5.70D-11	4	3.99D-07
12	9.52D-09	7	4.60D-07	5	9.75D-03
13	5.44D-07	8	1.99D-04	6	**
14	1.50D-05	9	1.04D-02		
15	2.26D-04	10	1.16D-01		
16	2.00D-03	11	**		
17	1.14D-02				
18	4.41D-02				

(127,64) BCH code

K	0dB	K	3dB	K	6dB
8	4.01D-13	4	*	3	4.79D-11
9	6.56D-11	5	5.60D-11	4	2.66D-05
10	4.45D-09	6	1.30D-07	5	1.59D-02
11	1.48D-07	7	2.90D-05	6	**
12	2.78D-06	8	1.25D-03		
13	3.21D-05	9	1.66D-02		
14	2.47D-04	10	9.39D-02		
15	1.35D-03	11	**		
16	5.55D-03				
17	1.77D-02				
18	4.57D-02				

(63,30) BCH code

K	0dB	K	3dB	K	6dB
6	1.78D-13	3	*	2	2.81D-08
7	2.41D-11	4	2.12D-11	3	1.64D-04
8	1.29D-09	5	3.11D-08	4	1.46D-02
9	3.44D-08	6	4.81D-06	5	1.50D-01
10	5.28D-07	7	1.74D-04	6	**
11	5.23D-06	8	2.32D-03		
12	3.62D-05	9	1.51D-02		
13	1.86D-04	10	5.85D-02		
14	7.50D-04	11	1.53D-01		
15	2.45D-03	12	**		
16	6.72D-03				
17	1.58D-02				
18	3.28D-02				

E_b/N_0=15dB, N=127

(1023,513) BCH code

K	0dB	K	3dB	K	6dB
14	*	7	*	4	*
15	5.11D-13	8	3.11D-13	5	1.40D-06
16	3.06D-09	9	1.85D-06	6	**
17	2.62D-06	10	1.10D-02		
18	3.92D-04	11	**		
19	1.33D-02				
20	1.30D-01				

(255,131) BCH code

K	0dB	K	3dB	K	6dB
12	1.93D-13	6	*	4	4.42D-11
13	7.24D-11	7	5.59D-11	5	1.76D-04
14	9.72D-09	8	4.55D-07	6	1.11D-01
15	5.54D-07	9	1.98D-04	7	**
16	1.53D-05	10	1.04D-02		
17	2.29D-04	11	1.16D-01		
18	2.02D-03	12	**		
19	1.14D-02				
20	4.44D-02				

(127,64) BCH code

K	0dB	K	3dB	K	6dB
10	4.10D-13	5	*	4	1.12D-07
11	6.68D-11	6	5.48D-11	5	1.16D-03
12	4.52D-09	7	1.29D-07	6	9.05D-02
13	1.50D-07	8	2.88D-05	7	**
14	2.81D-06	9	1.24D-03		
15	3.25D-05	10	1.66D-02		
16	2.50D-04	11	9.38D-02		
17	1.36D-03	12	**		
18	5.59D-03				
19	1.78D-02				
20	4.59D-02				

(63,30) BCH code

K	0db	K	3dB	K	6dB
8	1.80D-13	4	*	3	1.71D-11
9	2.44D-11	5	2.06D-11	4	4.35D-06
10	1.39D-09	6	3.06D-08	5	2.19D-03
11	3.48D-08	7	4.77D-06	6	5.67D-02
12	5.34D-07	8	1.73D-04	7	**
13	5.28D-06	9	2.31D-03		
14	3.65D-05	10	1.51D-02		
15	1.87D-04	11	5.84D-02		
16	7.55D-04	12	1.53D-01		
17	2.46D-03	13	**		
18	6.75D-03				
19	1.59D-02				
20	3.29D-02				

Lemma 3: The function $h(\hat{\tau}_i, \hat{\phi}_i) = E\,\{\exp\,[z(P_i/P_1)^{1/2}\hat{I}_i^0(\hat{\tau}_i, \hat{\phi}_i)]\}$ where $\hat{\tau}_i$ belongs to $[0, T_c/2]$, $\hat{\phi}_i$ belongs to $[0, \pi/2]$ and $z \geq 0$, increases as $\hat{\tau}_i$, or $\hat{\phi}_i$, or both, decrease.

The proof of Lemma 3 can be found in Appendix B. Due to Lemma 3, we have the following upper bound for the $\Pr(Y \geq 0)$:

$$\Pr(Y \geq 0) \leq \exp(-z)\,E\left[\exp(zn_0^*)\right] \cdot \prod_{i=2}^{K} E\left\{\exp\left[z(P_i/P_1)^{1/2}\hat{I}_i^0(0, \hat{\phi}_i)\right]\right\}. \quad (32)$$

It is worth noting that the value of the upper bound in (32) is independent of the specific choice $\hat{a}^{(1)}$ of the $a^{(1)}$ sequence [see (19)]. To simplify our notation we define $w_j(z, \hat{\phi}_j)$ as follows:

$$w_j(z, \hat{\phi}_j) = \left\{e^{-z}E\left[\exp(zn_0^*)\right]\right\}^{1/K-1} \cdot E\left\{\exp\left[z(P_j/P_1)^{1/2}\hat{I}_j^0(0, \hat{\phi}_j)\right]\right\}. \quad (33)$$

Then, from (14), (15), (16), (20), (32), and (33) we get

$$P_e(K) \leq \int_{\hat{\phi}} \sum_{i=e+1}^{M} \begin{bmatrix} M \\ i \end{bmatrix} \left[\prod_{j=2}^{K} w_j(z, \hat{\phi}_j)\right]^i \cdot \left[1 - \prod_{j=2}^{K} w_j(z, \hat{\phi}_j)\right]^{M-i} f_{\phi}(\hat{\phi})\, d\hat{\phi} \quad (34)$$

It is not difficult to see that the integrand in (34) is in a product form with respect to the variables $\hat{\phi}_j$ ($j = 2, \cdots, K$). Hence, to compute the RHS of (34) we need to evaluate single integrals instead of a multiple one. To be more specific let us write

$$\left[1 - \prod_{j=2}^{K} w_j(z, \hat{\phi}_j)\right]^{M-i} = \sum_{n=0}^{M-i} \begin{bmatrix} M-i \\ n \end{bmatrix} (-1)^n \left\{\prod_{j=2}^{K} w_j(z, \hat{\phi}_j)\right\}^n. \quad (35)$$

Due to (35), (34) becomes

$$P_e(K) \leq \sum_{i=e+1}^{M} \begin{bmatrix} M \\ i \end{bmatrix} \sum_{n=0}^{M-i} \begin{bmatrix} M-i \\ n \end{bmatrix} (-1)^n \cdot \left\{\prod_{j=2}^{K} \int_{\hat{\phi}_j} \left[w_j(z, \hat{\phi}_j)\right]^{i+n} f_{\phi_j}(\hat{\phi}_j)\, d\hat{\phi}_j\right\}. \quad (36)$$

TABLE III
COMPARISONS BETWEEN THE UPPER BOUNDS p_u AND q ON THE BIT ERROR PROBABILITY FOR EQUAL POWER SIGNALS

	$E_b/N_0=12$, N=31	
K	p_u	q
2	3.307D-04	3.245D-05
3	5.119D-03	5.950D-04
4	1.965D-02	2.567D-03
5	4.364D-02	6.227D-03
6	7.400D-02	1.133D-02
7	1.077D-01	1.748D-02
	$E_b/N_0=15$, N=31	
K	p_u	q
2	1.715D-05	1.522D-06
3	1.719D-03	1.865D-04
4	1.116D-02	1.385D-03
5	3.086D-02	4.230D-03
6	5.855D-02	8.678D-03
7	9.097D-02	1.437D-02
8	1.255D-01	2.091D-02

TABLE IV
THE UPPER BOUND $\hat{P}_e^u(K)$ ON THE PACKET ERROR PROBABILITY $P_e(K)$ FOR EQUAL POWER SIGNALS

K	$\hat{P}_e^u(K)$
3	*
4	*
5	*
6	*
7	*
8	2.51D-11
9	6.09D-07

TABLE V
COMPARISONS BETWEEN THE EXACT BIT ERROR PROBABILITY s AND THE UPPER BOUND p_u

N	E_b/N_0	K	NFR	s_l	s_u	p_u	r_l	r_u
15	12	2	0dB	3.543D-05	4.444D-05	4.592D-03	103	130
15	12	2	3dB	5.177D-04	6.430D-04	4.122D-02	64	80
15	12	2	6dB	4.254D-03	5.207D-03	1.757D-01	34	41
15	12	3	0dB	3.024D-04	4.430D-04	4.490D-02	101	148
15	12	3	3dB	2.827D-03	4.091D-03	1.813D-01	44	64
15	12	3	6dB	1.447D-02	2.073D-02	4.076D-01	20	28
15	15	2	0dB	4.411D-06	5.772D-06	1.211D-03	210	274
15	15	2	3dB	2.306D-04	2.915D-04	2.747D-02	94	119
15	15	2	6dB	3.158D-03	3.894D-03	1.577D-01	40	50
15	15	3	0dB	1.148D-04	1.712D-04	3.144D-02	183	273
15	15	3	3dB	1.925D-03	2.805D-03	1.639D-01	59	85
15	15	3	6dB	1.248D-02	1.792D-02	3.969D-01	22	32
31	12	2	0dB	2.156D-06	2.540D-06	3.307D-04	130	153
31	12	2	3dB	3.628D-05	4.281D-05	4.720D-03	110	130
31	12	2	6dB	5.118D-04	6.007D-04	4.084D-02	68	80
31	15	2	0dB	3.587D-08	4.377D-08	1.715D-05	392	478
31	15	2	3dB	5.433D-06	6.542D-06	1.448D-03	221	266
31	15	2	6dB	2.364D-04	2.800D-04	2.794D-02	100	118

TABLE VI
THE UPPER BOUND $\tilde{P}_e^u(K)$ ON THE PACKET ERROR PROBABILITY $P_e(K)$

	N=31, E_b/N_0=12, (63,30) BCH code		
K	0dB	3dB	6dB
2	-	-	2.36D-03
3	-	9.85D-04	-
4	3.43D-06	-	-
5	2.99D-04	-	-
	N=31, E_b/N_0=15, (63,30) BCH code		
K	0dB	3dB	6dB
2	-	-	2.57D-04
3	-	1.03D-04	-
4	5.59D-08	-	-
5	2.28D-05	-	-
6	9.41D-04	-	-

Note that $f_{\phi}(\hat{\phi})$ in (34) corresponds to the probability density function of the vector ϕ. Similarly $f_{\phi_j}(\hat{\phi}_j)$ is the probability density function of the phase of the jth interfering signal. Let us now denote by $\tilde{P}_e^u(K)$ the RHS of (36) when $z = z^*$ where z^* corresponds to the value of z which attains the infimum of the expression in formula (36). That is,

$$\tilde{P}_e^u(K) = \sum_{i=e+1}^{M} \begin{bmatrix} M \\ i \end{bmatrix} \sum_{n=0}^{M-i} \begin{bmatrix} M-i \\ n \end{bmatrix} (-1)^n \cdot \left\{ \prod_{j=2}^{K} \int_{\hat{\phi}_j} \left[w_j(z^*, \hat{\phi}_j) \right]^{i+n} f_{\phi_j}(\hat{\phi}_j)\, d\hat{\phi}_j \right\}. \quad (37)$$

Based on Lemma 3 we can compute arbitrarily tight upper bounds for the single integrals contained in expression (37). Hence, an arbitrarily tight upper bound of $\tilde{P}_e^u(K)$ can be evaluated. It turns out that this is a computationally intensive procedure due, primarily, to the fact that the number of points required to compute the integrals in (37) is large. Furthermore, the computational complexity of the task to evaluate $\tilde{P}_e^u(K)$ with accuracy increases rapidly as the code length (M) increases.

In Table VI, we show tight upper bounds of $\tilde{P}_e^u(K)$ for the (63, 30) BCH code and for the same parameter values (i.e., K, E_b/N_0, N, near-far ratios) considered in Table I.

The results in Table VI allow us to make certain observations.

O.6) $\tilde{P}_e^u(K)$ is between an order and two orders of magnitude better than $P_e^u(K)$.

O.7) The improvement achieved by $\tilde{P}_e^u(K)$ decreases as the near-far ratio increases.

Similar results as the ones presented in Table VI were obtained for the (127, 64) BCH code. The main reason that motivated us to present the above improved upper bound on the packet error probability $P_e(K)$ is to get a quantitative idea of how good an upper bound the Chernoff bounding technique (see Lemma 2) gives us.

V. CONCLUSION

We presented an upper bound on the packet error probability induced in direct-sequence spread-spectrum networks. Furthermore, we compared our upper bound to already existing results, we examined its tightness, and we discussed one way of improving it.

An important advantage of the bound p_u, derived in Section III, is that its validity (see Lemmas 1 and 2 and Appendix B) does not rely on any assumptions about the joint probability density function $f_{\tau,\phi}$ of all delays and phases (e.g., independence of the delays and phases). The only assumption that we used for the derivation of p_u is that each delay and phase includes in its range the zero value. Furthermore, the form of the bound p_u is simple and easily computable [see (23)]. Once p_u is calculated the computation of $P_e^u(K)$ especially for BCH codes [see (25)] becomes a straightforward task. More importantly, our presentation is Section III has shown that the upper bound on the packet error probability is valid independently of whether signals arrive with equal or unequal powers at the receiver site.

Appendix A

For $N = 2$, $K = 3$, $\phi_2 = \phi_3 = 0$, $\tau_2 = \tau_3 = T_c/2$, and $P_2/P_1 = P_3/P_1 = 1$ we have that [see formula (9)]

$$X_0 = n_0^* + 1 + \left[a_{-1}^{(2)} a_0^{(1)} + a_0^{(2)} a_0^{(1)} + a_1^{(2)} a_1^{(1)} + a_0^{(2)} a_1^{(1)} + a_{-1}^{(3)} a_0^{(1)} + a_0^{(3)} a_0^{(1)} + a_1^{(3)} a_1^{(1)} + a_0^{(3)} a_1^{(1)} \right] /4 \quad (A.1)$$

$$X_1 = n_1^* + 1 + \left[a_1^{(2)} a_2^{(1)} + a_2^{(2)} a_2^{(1)} + a_3^{(2)} a_3^{(1)} + a_2^{(2)} a_3^{(1)} + a_1^{(3)} a_2^{(1)} + a_2^{(3)} a_2^{(1)} + a_3^{(3)} a_3^{(1)} + a_2^{(3)} a_3^{(1)} \right] /4. \quad (A.2)$$

Then,

$$\begin{aligned} \Pr\left(X_0 < 0 \cap X_1 < 0 \right) &= 1/4 \Pr\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = 1,\, a_1^{(3)} = 1 \right) \\ &+ 1/4 \Pr\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = -1,\, a_1^{(3)} = -1 \right) \\ &+ 1/4 \Pr\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = 1,\, a_1^{(3)} = -1 \right) \\ &\cdot 1/4 \Pr\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = -1,\, a_1^{(3)} = 1 \right). \end{aligned} \quad (A.3)$$

By direct substitution we can see that

$$\begin{aligned} \Pr&\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = 1,\, a_1^{(3)} = 1 \right) \\ &= \Pr\left[\left\{ n_0^* + 1 + a_0^{(1)} \left(a_{-1}^{(2)} + a_0^{(2)} + a_{-1}^{(3)} + a_0^{(3)} \right) + a_1^{(1)} \left(2 + a_0^{(2)} + a_0^{(3)} \right) < 0 \right\} \cap \left\{ n_1^* + 1 + a_3^{(1)} \left(a_3^{(2)} + a_2^{(2)} + a_3^{(3)} + a_2^{(3)} \right) + a_2^{(1)} \left(2 + a_2^{(2)} + a_2^{(3)} \right) < 0 \right\} \right]. \end{aligned} \quad (A.4)$$

Let $X_0^{(+,+)}$ be a random variable equal to X_0 with $a_1^{(2)} = a_1^{(3)} = 1$. Then, we deduce from (A.4) that

$$\Pr\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = 1,\, a_1^{(3)} = 1 \right) = \left[\Pr\left(X_0^{(+,+)} < 0 \right) \right]^2. \quad (A.5)$$

Also,

$$\begin{aligned} \Pr&\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = -1,\, a_1^{(3)} = -1 \right) \\ &= \Pr\left[\left\{ n_0^* + 1 + a_0^{(1)} \left(a_1^{(2)} + a_0^{(2)} + a_{-1}^{(3)} + a_0^{(3)} \right) + a_1^{(1)} \left(-2 + a_0^{(2)} + a_0^{(3)} \right) < 0 \right\} \cap \left\{ n_1^* + 1 + a_3^{(1)} \left(a_3^{(2)} + a_2^{(2)} + a_3^{(3)} + a_2^{(3)} \right) + a_2^{(1)} \left(-2 + a_2^{(2)} + a_2^{(3)} \right) < 0 \right\} \right]. \end{aligned} \quad (A.6)$$

We can write (A.6) as follows:

$$\begin{aligned} \Pr&\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = -1,\, a_1^{(3)} = -1 \right) \\ &= \Pr\left[\left\{ n_0^* + 1 - a_0^{(1)} \left(-a_{-1}^{(2)} - a_0^{(2)} - a_{-1}^{(3)} - a_0^{(3)} \right) - a_1^{(1)} \left(2 - a_0^{(2)} - a_1^{(3)} \right) < 0 \right\} \cap \left\{ n_1^* + 1 - a_3^{(1)} \left(-a_3^{(2)} - a_2^{(2)} - a_3^{(3)} - a_2^{(3)} \right) - a_2^{(1)} \left(2 - a_2^{(2)} - a_2^{(3)} \right) < 0 \right\} \right]. \end{aligned} \quad (A.7)$$

Every $-a$ item in (A.7) is a binary random variable assuming values $+1$ or -1 with equal probability. Hence,

$$\Pr\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = -1,\, a_1^{(3)} = -1 \right) = \left[\Pr\left(X_0^{(+,+)} < 0 \right) \right]^2. \quad (A.8)$$

We can work similarly to show that

$$\begin{aligned} \Pr\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = 1,\, a_1^{(3)} = -1 \right) &= \Pr\left(X_0 < 0 \cap X_1 < 0 / a_1^{(2)} = -1,\, a_1^{(3)} = 1 \right) \\ &= \left[\Pr\left(X_0^{(+,-)} < 0 \right) \right]^2 \end{aligned} \quad (A.9)$$

where $X_0^{(+,-)}$ is equal to X_0 with $a_1^{(2)} = 1$ and $a_1^{(3)} = -1$.

From (A.3), (A.5), (A.8), and (A.9) we take

$$\Pr\left(X_0 < 0 \cap X_1 < 0 \right) = \frac{1}{2} \left[\Pr\left(X_0^{(+,+)} < 0 \right) \right]^2 + \frac{1}{2} \left[\Pr\left(X_0^{(+,-)} < 0 \right) \right]^2. \quad (A.10)$$

Furthermore,

$$\Pr\left(X_0 < 0 \right) = \Pr\left(X_1 < 0 \right) = \frac{1}{2} \left[\Pr\left(X_0^{(+,+)} < 0 \right) + \Pr\left(X_0^{(+,-)} < 0 \right) \right]. \quad (A.11)$$

Equations (A.10) and (A.11) allow us to state that

$$\Pr\left(X_0 < 0 \cap X_1 < 0 \right) = \Pr\left(X_0 < 0 \right) \Pr\left(X_1 < 0 \right) \quad (A.12)$$

if and only if

$$\Pr\left(X_0^{(+,+)} < 0 \right) = \Pr\left(X_0^{(+,-)} < 0 \right). \quad (A.13)$$

After some algebra, we showed that

$$\begin{aligned} \Pr\left(X_0^{(+,+)} < 0 \right) &= 1/4 \big[1/16 Q(-d) + 2/16 Q(-0.5d) + 8/16 Q(0) \\ &+ 14/16 Q(0.5d) + 14/16 Q(d) + 14/16 Q(1.5d) \\ &+ 8/16 Q(2d) + 2/16 Q(2.5d) \\ &+ 1/16 Q(3d) \big] \end{aligned} \quad (A.14)$$

where

$$d = -\left(2 E_b / N_0 \right)^{1/2} \quad (A.15)$$

and

$$Q(x) = (2\pi)^{-1/2} \int_x^{\infty} \exp\left(-y^2/2 \right) dy \quad (A.16)$$

$$\begin{aligned} \Pr\left(X_0^{(+,-)} < 0 \right) &= 1/4 \big[2/16 Q(-0.5d) + 4/16 Q(0) + 14/16 Q(0.5d) \\ &+ 24/16 Q(d) + 14/16 Q(1.5d) + 4/16 Q(2d) \\ &+ 2/16 Q(2.5d) \big] \end{aligned} \quad (A.17)$$

with d and Q defined in (A.15) and (A.16).

Consequently,

$$\Pr\left(X_0^{(+,+)} < 0 \right) - \Pr\left(X_0^{(+,-)} < 0 \right) = 1/4 \big[3/16 + 4/16 Q(2d) + 1/16 Q(3d) - 11/16 Q(d) \big]. \quad (A.18)$$

By choosing E_b/N_0 large enough, we can make the above difference negative. Hence, in general,

$$\Pr\left(X_0^{(+,+)} < 0 \right) \neq \Pr\left(X_0^{(+,-)} < 0 \right)$$

or equivalently

$$\Pr\left(X_0 < 0 \cap X_1 < 0 \right) \neq \Pr\left(X_0 < 0 \right) \Pr\left(X_1 < 0 \right).$$

Appendix B

Let us examine the term $E\{\exp[z_i \hat{I}_i^0(\hat{\tau}_i, \hat{\phi}_i)]\}$ where z_i stands for $z(P_i/P_1)^{1/2}$. We first define S_i^A to be the set of sequences $a^{(i)}$ such that $a_{-1}^{(i)} \hat{a}_0^{(1)} = -a_{N-1}^{(i)} \hat{a}_{N-1}^{(1)}$ and S_i^B the set of sequences $a^{(i)}$

such that $a_{-1}^{(i)}\hat{a}_0^{(1)} = a_{N-1}^{(i)}\hat{a}_{N-1}^{(1)}$. Then, from the total probability formula we get

$$E\{\exp[z_i\hat{I}_i^0(\hat{\tau}_i,\hat{\phi}_i)]\} = \sum_{a^{(i)}\in S_i^A} E\{\exp[z_i\hat{I}_i^0(\hat{\tau}_i,\hat{\phi}_i)] \mid a^{(i)}\in S_i^A\}\Pr(a^{(i)}\in S_i^A) + \sum_{a^{(i)}\in S_i^B} E\{\exp[z_i\hat{I}_i^0(\hat{\tau}_i,\hat{\phi}_i)] \mid a^{(i)}\in S_i^B\}\Pr(a^{(i)}\in S_i^B). \tag{B.1}$$

We concentrate on the terms $E\{\exp[z_i\hat{I}_i^0(\hat{\tau}_i,\hat{\phi}_i)]/a^{(i)}\in S_i^A\}$ first. It is easy to show [see (17)] that

$$E\{\exp[z_i\hat{I}_i^0(\hat{\tau}_i,\hat{\phi}_i)] \mid a^{(i)}\in S_i^A\} = E\{\exp[z_i\hat{I}_{i,A}^0(\hat{\tau}_i,\hat{\phi}_i)]\} \tag{B.2}$$

where

$$\hat{I}_{i,A}^0(\hat{\tau}_i,\hat{\phi}_i) = \left[(1-2\hat{\tau}_i/T_c)\sum_{j\in C} a_j^{(i)}\hat{a}_j^{(1)} + \sum_{j\in D} a_j^{(i)}\hat{a}_j^{(1)} + (1-2\hat{\tau}_i/T_c)a_{N-1}^{(i)}\hat{a}_{N-1}^{(1)}\right]\cos\hat{\phi}_i/N \tag{B.3}$$

with

$$C = \{j:\ \hat{a}_j^{(1)} \neq \hat{a}_{j+1}^{(1)};\ 0\le j\le N-2\} \tag{B.4}$$

$$D = \{j:\ \hat{a}_j^{(1)} = \hat{a}_{j+1}^{(1)};\ 0\le j\le N-2\}. \tag{B.5}$$

The form of $\hat{I}_{i,A}^0(\hat{\tau}_i,\hat{\phi}_i)$ indicates that we should only consider $\hat{\tau}_i\in[0, T_c/2]$ and $\hat{\phi}_i\in[0,\pi/2]$ (see [5, Appendix, p. 10] for more details).

For $\hat{\phi}_i\in[0,\pi/2]$ and $\hat{\tau}_{i,1}\le\hat{\tau}_{i,2}$, such that $\hat{\tau}_{i,1},\hat{\tau}_{i,2}\in[0,T_c/2]$, we will show that

$$E\{\exp[z_i\hat{I}_{i,A}^0(\hat{\tau}_{i,2},\hat{\phi}_i)]\} \le E\{\exp[z_i\hat{I}_{i,A}^0(\hat{\tau}_{i,1},\hat{\phi}_i)]\}. \tag{B.6}$$

Let us first define the random variables U and X

$$U = \sum_{j\in D} a_j^{(i)}\hat{a}_j^{(1)} \tag{B.7}$$

$$X = \sum_{j\in C} a_j^{(i)}\hat{a}_j^{(1)} + a_{N-1}^{(i)}\hat{a}_{N-1}^{(1)}. \tag{B.8}$$

Then, we can write

$$\begin{aligned} E\{[\exp[z_i\hat{I}_{i,A}^0(\hat{\tau}_i,\hat{\phi}_i)]\} &= \sum_u E\{\exp[z_i,\hat{I}_{i,A}^0(\hat{\tau}_i,\hat{\phi}_i)] \mid U=u\}\Pr[U=u] \\ &= \sum_u \exp(u\cos\hat{\phi}_i/N)E\{[\exp(z_iX(1-2\hat{\tau}_i/T_c) \\ &\quad \cdot\cos\hat{\phi}_i/N)]\}\Pr[U=u] \end{aligned} \tag{B.9}$$

where $\hat{\tau}_i$ stands for $\hat{\tau}_{i,1}$ or $\hat{\tau}_{i,2}$. Furthermore, we can show that

$$E\{[\exp(z_iX(1-2\hat{\tau}_{i,2}/T_c)\cos\hat{\phi}_i/N)]\} \le E\{[\exp(z_iX(1-2\hat{\tau}_{i,1}/T_c)\cos\hat{\phi}_i/N)]\}. \tag{B.10}$$

Inequality (B.10) has the following form:

$$\begin{aligned} &\sum_{x\ge 0}[\exp(z_ix(1-2\hat{\tau}_{i,2}/T_c)) + \exp(-z_ix(1-2\hat{\tau}_{i,2}/T_c))]\Pr[X=x] \\ &\le \sum_{x\ge 0}[\exp(z_ix(1-2\hat{\tau}_{i,1}/T_c)) + \exp(-z_ix(1-2\hat{\tau}_{i,1}/T_c))]\Pr[X=x]. \end{aligned} \tag{B.11}$$

(B.11) is true due to the increasing nature of the function $h(t) = e^t + e^{-t}$ $(t\ge 0)$. (B.9) and (B.10) prove the validity of (B.6).

Working similarly as above, we can also show that for $\hat{\tau}_i\in[0, T_c/2]$ and $\hat{\phi}_{i,1}\le\hat{\phi}_{i,2}$, such that $\hat{\phi}_{i,1},\hat{\phi}_{i,2}\in[0,\pi/2]$, the following inequality is true

$$E\{\exp[z_i\hat{I}_{i,A}^0(\hat{\tau}_i,\hat{\phi}_{i,2})]\} \le E\{\exp[z_i,\hat{I}_{i,A}^0(\hat{\tau}_i,\hat{\phi}_{i,1})]\}. \tag{B.12}$$

Combining (B.6) and (B.12) we conclude that $E\{\exp[z_i\hat{I}_{i,A}^0(\hat{\tau}_i,\hat{\phi}_i)]\}$ increases as $\hat{\tau}_i$, or $\hat{\phi}_i$, or both decrease. In addition to that, we can show that $E\{\exp[z_i\hat{I}_{i,B}^0(\hat{\tau}_i,\hat{\phi}_i)]\}$ increases as $\hat{\tau}_i$, or $\hat{\phi}_i$, or both decrease where

$$E\{\exp[z_i\hat{I}_{i,B}^0(\hat{\tau}_i,\hat{\phi}_i)]\} = E\{\exp[z_i\hat{I}_i^0(\hat{\tau}_i,\hat{\phi}_i) \mid a^{(i)}\in S_i^B\} \tag{B.13}$$

with

$$\hat{I}_{i,B}^0(\hat{\tau}_i,\hat{\phi}_i) = \left\{(1-2\hat{\tau}_i/T_c)\sum_{j\in C} a_j^{(i)}\hat{a}_j^{(1)} + \sum_{j\in D} a_j^{(i)}\hat{a}_j^{(1)} + a_{N-1}^{(i)}\hat{a}_{N-1}^{(1)}\right\}\cos\hat{\phi}_i/N. \tag{B.14}$$

The above discussion and formula (B.1) proves the validity of Lemma 3.

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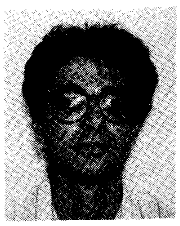

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